An Attitude Reference System with Discrete-Correction Capability

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For long-term space missions, redundancy in both hardware and software is often desired. This paper describes an attitude reference system that possesses a discrete correction capability, as well as several measures of redundancy. Ten basic product-variables are defined, where these variables are related to the four Euler quaternion parameters. While the Euler normalization condition is quadratic in the Euler parameters, it becomes linear in the product-variables. As a direct result, it is possible to exactly satisfy the normalization condition for all time even in the presence of round-off errors. A numerical example is provided to illustrate the discrete correction capability. Relevant identities and differential equations are summarized.

Nomenclature

A_{ij}	= direction cosine matrix elements
e_i	= Euler quaternion parameters
N	= number of integration steps to complete one orbit
N_s	= step number for numerical integration
Q	= normalization constant (error-free value unity)
T	= orbit period
w_{ij}	= product-variables (16 total, 10 basic)
x, y, z	= body axis orthogonal coordinates
X, Y, Z	= inertially fixed orthogonal coordinate system
y	= general integration variable
α, β, γ	= direction angles for Euler axis
Δt	= integration step size
Δ_{ij}	= orthogonalization quantities (error-free value zero)
θ	= Euler rotation angle
$\omega_x, \omega_y, \omega_z$	= body rates in orthogonal body coordinates

Introduction

FOR long-term space missions the requirements placed on navigation and attitude control are extremely severe. (Some effects of error sources on navigation accuracies have been given by Jordan, Madrid, and Pease.[1]) Increasingly greater emphasis is placed on redundancy, error-detection, and error-correction capabilities. While the effect of these requirements is most clearly manifested in the hardware, the basic redundancy should, so far as feasible, be carried into all aspects of the system.

The present paper defines an attitude reference system that is highly redundant. This system should be used in conjunction with a gimballess inertial measurement unit. It is noted that the choice of an attitude reference system for a particular application typically depends on the basic rates of attitude change which are anticipated, on control system requirements, and on properties of the digital computer (e.g., word length and computation speed), together with basic accuracy requirements on pointing. Traditionally, Euler angles have found widespread usage (for example, in aircraft attitude reference, and also on the Apollo-Saturn booster where an IMU is employed). More recently, for the Skylab program, the Euler four-parameter system is to be employed. The product-variable reference system, which is defined in the present paper, is closely related to the Euler four-parameter description.

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Euler Four-Parameter System

Consider an inertially fixed coordinate system X, Y, Z, and consider a body free to move in space. According to Euler's theorem,[2] if a finite attitude change occurs, it is possible to express this attitude change by a single rotation about a fixed axis in the body (say the x axis), where this axis is defined by the angles α, β, γ between (x, X), (x, Y), and (x, Z), respectively (Fig. 1). The rotation angle θ, together with the previous three direction-cosine angles, then leads to four basic parameters that define the attitude change for rotation about the x axis.

$$e_1 = \cos(\theta/2) \tag{1}$$

$$e_2 = \cos\gamma \sin(\theta/2) \tag{2}$$

$$e_3 = \cos\beta \sin(\theta/2) \tag{3}$$

$$e_4 = \cos\alpha \sin(\theta/2) \tag{4}$$

If one now considers a body axis system, x, y, z, that initially coincides with X, Y, Z, the transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{5}$$

relates the moving (body) axes to the inertial axes. The equations relating the A_{ij} elements to the quaternion e_i parameters may be found in various texts.[2] It is now noted, however, that each expression of the form

$$A_{ij} = F(e_i, e_j) \tag{6}$$

involves sums and differences of quadratic terms in the e_i parameters. As an example, a typical diagonal, and an off-diagonal element are expressed by

$$A_{11} = e_1^2 - e_2^2 - e_3^2 + e_4^2 \tag{7}$$

$$A_{12} = 2(e_1 e_2 + e_3 e_4) \tag{8}$$

It is also recalled that the normality condition for the Euler parameters involves a quadratic expression, where explicitly

$$Q = \sum_{i=1}^{4} e_i^2 = 1 \tag{9}$$

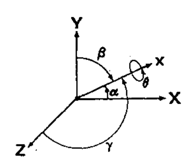

Fig. 1 Four parameter geometry.

Finally, it is noted that the time rates of change of the Euler parameters are related to body rates ω_x, ω_y, and ω_z by the equations

$$2\dot{e}_1 = -\omega_z e_2 - \omega_y e_3 - \omega_x e_4 \tag{10}$$

$$2\dot{e}_2 = \omega_z e_1 - \omega_x e_3 + \omega_y e_4 \tag{11}$$

$$2\dot{e}_3 = \omega_y e_1 + \omega_x e_2 - \omega_z e_4 \tag{12}$$

$$2\dot{e}_4 = \omega_x e_1 - \omega_y e_2 + \omega_z e_3 \tag{13}$$

Comments on the Four-Parameter System

As an initial observation, it is noted that a primary advantage of Euler parameters is indicated by the simplicity of Eqs. (10–13), where for a given set of instantaneous body rates ω_x, ω_y, and ω_z, the numerical integration is quite straightforward. Morover, because trigonometric functions do not appear in Eqs. (5–13), a fast computation cycle is readily obtained. While an actual resolution of "pointing angles" would require inverse trigonometric operations, an attitude reference system utilizing Euler parameters is generally structured to require minimal use of Eqs. (1–4) and their inverse.

Regarding some potential numerical difficulties with the four-parameter system, it is first observed that Eqs. (10–13), when expressed in matrix form, represent a highly off-diagonal or skewed system of differential equations. (Note for example, that in Eq. (10), the term $\dot{e}_1$ is coupled to e_2, e_3 and e_4, but not to e_1.) This situation should be contrasted with the strong-diagonal case, which is most desirable for minimizing truncation errors. Numerical simulation results typically indicate that a long term drift can arise, and in order to reduce this drift it is common to renormalize the e_i parameters by dividing by the quantity $Q^{1/2}$ of Eq. (9) after each update of the four parameters. Although this procedure results in some reduction of long-term drift, it is nevertheless noted than an error ε_i in a parameter e_i is multiplied by a factor of two in forming e_i^2. Thus, the normalization quantity Q of Eq. (9) is itself subject to varying accuracy.

Product-Variable Reference System

The exact satisfaction of Eq. (9) furnishes the motivation for defining the product-variables:

$$w_{ij}(t) = e_i(t)e_j(t) \tag{14}$$

where $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$. In terms of the w_{ij} variables, the normalization quantity is written:

$$Q = \sum_{i=1}^{4} w_{ii}(t) \tag{15}$$

It is also noted that the various products (e_ie_j) of Eq. (6) can be represented by single w_{ij} variables. The special role of the w_{ij} variables and normality will be discussed presently. Meanwhile, the relevant equations for the product-variables may now be derived. It is immediately apparent that, while there formally exist 16 product-variables, the symmetry relation

$$w_{ij}(t) = w_{ji}(t) \tag{16}$$

permits a reduction to ten basic product-variables. Hereafter these basic variables will be specified: $w_{11}, w_{12}, w_{13}, w_{14}, w_{22}, w_{23}, w_{24}, w_{33}, w_{34}$ and w_{44}. In terms of these ten variables, the identity

$$w_{ij}(t)^2 = w_{ii}(t)w_{jj}(t) \tag{17}$$

can be used to derive six relationships between the previously specified variables.

In obtaining the differential equations for $\dot{w}_{ij}(t)$, it is first noted that

$$\dot{w}_{ij} = e_i\dot{e}_j + \dot{e}_ie_j \tag{18}$$

and, of course, with $i = j$,

$$\dot{w}_{ij} = 2e_i\dot{e}_i \tag{19}$$

The equations for $\dot{w}_{ii}$ are obtained by multiplying through Eq. (10–13) by e_1, e_2, e_3 and e_4, respectively. It then follows that

$$\dot{w}_{11} = -\omega_z w_{12} - \omega_y w_{13} - \omega_x w_{14} \tag{20}$$

$$\dot{w}_{22} = \omega_z w_{12} - \omega_x w_{23} + \omega_y w_{24} \tag{21}$$

$$\dot{w}_{33} = \omega_y w_{13} + \omega_x w_{23} - \omega_z w_{34} \tag{22}$$

$$\dot{w}_{44} = \omega_x w_{14} - \omega_y w_{24} + \omega_z w_{34} \tag{23}$$

The equations for $\dot{w}_{ij}$ where $i \neq j$ are similarly obtained. (For example, the $\dot{w}_{12}$ equation is derived by multiplying Eq. (10) by e_2 and Eq. (11) e_1, then adding the resulting expressions.) Finally, the $\dot{w}_{ij}$ equations may be written

$$2\dot{w}_{12} = -\omega_x(w_{24} + w_{13}) + \omega_y(w_{14} - w_{23}) + \omega_z(w_{11} - w_{22}) \tag{24}$$

$$2\dot{w}_{13} = \omega_x(w_{12} - w_{34}) + \omega_y(w_{11} - w_{33}) - \omega_z(w_{14} + w_{23}) \tag{25}$$

$$2\dot{w}_{14} = \omega_x(w_{11} - w_{44}) - \omega_y(w_{12} + w_{34}) + \omega_z(w_{13} - w_{24}) \tag{26}$$

$$2\dot{w}_{23} = \omega_x(w_{22} - w_{33}) + \omega_y(w_{12} + w_{34}) + \omega_z(w_{13} - w_{24}) \tag{27}$$

$$2\dot{w}_{24} = \omega_x(w_{12} - w_{34}) + \omega_y(w_{44} - w_{22}) + \omega_z(w_{14} + w_{23}) \tag{28}$$

$$2\dot{w}_{34} = \omega_x(w_{13} + w_{24}) + \omega_y(w_{14} - w_{23}) + \omega_z(w_{33} - w_{44}) \tag{29}$$

Invariant Property of the Normality Condition (9)

Perhaps the most striking feature obtained from the use of the product-variable reference system is the exact preservation of Eq. (9) for all time t, where Q is given by Eq. (15). Inasmuch as most aerospace computers do not have a "floating arithmetic" capability, attention will be restricted (throughout the present discussion) to fixed binary calculations.

The mechanism for the loss of normality in the four-parameter system should first be identified before proceeding with the w_{ij} system. In particular, suppose attention is placed on three specific body rates ω_x, ω_y, and ω_z, and suppose that at some instant in time, $t = t_0$, values are given for $e_i(t_0)$ such that Eq. (9) is satisfied to the lowest-order bit. In the course of forming the products on the right side of Eqs. (10–13), it is noted that each of the products is nonrepetitive, i.e., the products are unique and typically nonequal. In consequence, round-off errors provide a mechanism for the violation of Eq. (9) at some time t_1, where $t_1 > t_0$. The extent to which Eq. (9) is violated on a single integration step depends on the direction of the respective round-off errors. A more serious consideration arises when one considers several steps of integration and notes that there is no feedback or self-corrective mechanism in Eqs. (10–13) to reduce round-off errors once they occur.

Consider now the w_{ii} variables of Eqs. (20–23), where again specific body rates ω_x, ω_y, and ω_z are prescribed at time $t = t_0$, and where values for $w_{ii}(t_0)$ are prescribed such that Eq. (9) is satisfied to the lowest-order bit. It is now noted that each of the respective products on the right side of Eqs. (20–23) occurs exactly twice, where for each product there exists a corresponding negated product. In consequence, regardless of round-off or quantization effects, if a given product has a certain bit configuration, the negated product will have that same bit configuration with an opposite sign.‡ Hence, on adding Eqs. (20–23), it follows that

$$\sum_{i=1}^{4} \dot{w}_{ii} = 0 \tag{30}$$

where this relation is valid to the lowest-order bit. It then follows from Eqs. (9, 15, and 30) that if the normality condition is satisfied to the lowest-order bit at some time $t = t_0$, then it will continue to be satisfied for all time $t > t_0$.

A final comment should be made regarding the design of the numerical integrator. It is essential that the actual computer operations should be carefully ordered so that the positive-negative product-duality is preserved. Although this is easily accomplished, it may require some attention to detailed coding. If one wishes to use a high-order integrator, expressions for derivatives $w_{ij}^{(2)}(t)$, $w_{ij}^{(3)}(t)$, etc., can be obtained by direct differentiation of the relations (20–29).

Discrete-Correction Capability

The discrete-correction capability is an unusual feature arising from the exact satisfaction of the normality condition (9). The detailed structure of a discrete error condition will be illustrated presently for a specific example, but it should now be noted that $w_{ii}(t)$ variables must individually satisfy the conditions

$$0 \le w_{ii}(t) \le 1 \tag{31}$$

for $i = 1, 2, 3, 4$ for all time t. While it is equally true that, in the four-parameter system

$$-1 \le e_i(t) \le 1 \tag{32}$$

it is not generally possible to utilize Eq. (32) in a meaningful way due to violation of the normalization condition (9). As previously mentioned, Eq. (32) is generally forced into compliance by normalizing the e_i variables by $Q^{1/2}$ after each update of the four-parameters. While truncation and round-off errors in the individual w_{ii} variables can also cause Eq. (31) to appear to be violated, it will be seen that a rather unique error condition arises; it is this condition that provides a discrete-correction capability.

Numerical Results

Since the Euler parameters are to be used on Skylab, and since the w_{ij} variables are closely allied to Euler parameters, it was decided to choose a simple, but realistic example for demonstrating the product-variable behavior. In particular, for the Z-local-vertical mode, the Skylab vehicle is to maintain its Z-axis toward the Earth while circling the Earth at a nominal altitude of 220 naut miles. It is readily verified that the orbit rate is given by

$$|\bar{\omega}| = 0.064716 \text{ deg/sec} \tag{33}$$

where

$$\bar{\omega} = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k} \tag{34}$$

In the nominal case the attitude change will occur only about the $\bar{j}$ vector, and hence

$$\omega_x = 0 \tag{35}$$

$$\omega_y = 0.064716 \text{ deg/sec} \tag{36}$$

$$\omega_z = 0 \tag{37}$$

‡The author is indebted to the reviewer for noting that this statement should be given a functional interpretation rather than strictly interpreted in terms of computer hardware bit configuration. Two's complement arithmetic is functionally compatible.

The strapdown platform is initialized according to the conditions at $t = 0$,

$$w_{11}(0) = 0 \tag{38}$$

$$w_{22}(0) = 0 \tag{39}$$

$$w_{33}(0) = 1 \tag{40}$$

$$w_{44}(0) = 0 \tag{41}$$

The orbit period is given by

$$T = 5562.8 \text{ sec} \tag{42}$$

Regarding the computer program, the simulation was written in *PL/I*, using the fixed binary capability. The binary word consists of 31 bits, plus the sign bit, where the low-orbit bit represents $2^{-30} = 9.3(10)^{-10}$. The numerical integrator required for Eqs. (20–29) is chosen to be predictive (but not corrective), and is described by

$$y(t_1) = y(t_0) + y'(t_0)\Delta t + (y''(t_0)/2)(\Delta t)^2 \tag{43}$$

where $\Delta t = t_1 - t_0$. The Δt step size is computed from

$$\Delta t = (T/N) \tag{44}$$

where $N = 20{,}000$, and where T is given by Eq. (42).

It readily follows from conditions (38–41) that the relevant nonzero w_{ij} variables are w_{11}, w_{13}, and w_{33}. For the situation where the nominal constant rate (36) is exactly maintained, it is possible to express the w_{ij} variables in the analytic form

$$w_{11} = \tfrac{1}{2}[1 - \cos(\omega_y t)] \tag{45}$$

$$w_{13} = -\tfrac{1}{2}\sin(\omega_y t) \tag{46}$$

$$w_{33} = \tfrac{1}{2}[1 + \cos(\omega_y t)] \tag{47}$$

The general behavior of these variables is indicated in Fig. 2. Detailed numerical results for the case $N = 20{,}000$ are given in Table 1, where N_s denotes the integration step number. The half-orbit and full-orbit N_s values are given by (10,000, 20,000).

Referring to the $N = 20{,}000$ case, the computer output indicates that a violation of Eq. (31) first occurs at the step value $N_s = 9994$, and continues to occur until the value $N_s = 10{,}006$ is reached. A similar violation occurs (near the full orbit) at $N_s = 19{,}992$, and ceases occurring at $N_s = 20{,}008$. Referring to Table 1, a discrete correction is clearly desired near the peak values $N_s = 10{,}000$ and $N_s = 20{,}000$, where both w_{11} and w_{33} undergo their maximum excursion beyond the theoretical bounds of Eq. (31). If this correction is applied by interpolating for the maximum excursion, based on the three tabulated half-orbit values of w_{11} and w_{33}, the corrected values of w_{11} and w_{33} at $N_s = 10{,}000$ will possess a resulting error of less than $5(10)^{-8}$. Immediately subsequent to the discrete correction, the cross-product variable w_{13} should be corrected by application of Eq. (17).

Comparison of Euler Parameter and Product-Variable Results

It is interesting to compare the half-orbit and full-orbit values of w_{ij} previously presented with the corresponding Euler parameter results. In particular, Eqs. (10–13) can be numerically integrated subject to the conditions described by Eqs. (36, 42, 43 and 44), where initially

$$e_1(0) = 0 \tag{48}$$

$$e_2(0) = 0 \tag{49}$$

$$e_3(0) = 1 \tag{50}$$

$$e_4(0) = 0 \tag{51}$$

Corresponding to the half-orbit and full-orbit values $N_s = 10{,}000$ and $N_s = 20{,}000$ of Table 1, the Euler parameters e_1 and e_3 are given by

a) $N_s = 10{,}000$

$$e_1 = -0.999997045 \quad (52)$$

$$e_3 = 0.000000887 \quad (53)$$

b) $N_s = 20{,}000$

$$e_1 = -0.000001828 \quad (54)$$

$$e_3 = -0.999987436 \quad (55)$$

The normalization quantity Q of Eq. (9) may be directly evaluated at the half-orbit and full-orbit positions with the result that

$$Q = 0.999994090 \quad (56)$$

for $N_s = 10{,}000$, and

$$Q = 0.999974872 \quad (57)$$

for $N_s = 20{,}000$. Both of these quantities differ from the theoretical value of unity. In contrast, it is noted that the sum of w_{11} and w_{33} in Table 1 exactly maintains a unit value. This latter result explicitly demonstrates the control of round-off error in the normalization. It should not, of course, be implied that w_{11} and w_{33} are individually free of round-off error.

A simple comparison of the Euler parameter and product-variable accuracy is readily obtained by applying Eq. (14) to the results (52–57). If, in each case, attention is placed on the value e_i having the greatest number of significant figures, i.e., the fewest leading zeros, it follows that Eqs. (52) and (55) imply for the half-orbit and full-orbit values

$$w_{11} = 0.999994090 \quad (58)$$

$$w_{33} = 0.999974872 \quad (59)$$

The differences from the theoretical value of unity are then given by

$$|\Delta w_{11}| = 6(10)^{-6} \quad (60)$$

$$|\Delta w_{33}| = 3(10)^{-5} \quad (61)$$

Table 1 Numerical results for $N = 20{,}000$

N_s	w_{11}	w_{13}	w_{33}
	One-half orbit		
9,999	1.000000852	−0.000085	−0.000000852
10,000	1.000000854	0.000073	−0.000000854
10,001	1.000000807	0.000230	−0.000000807
	One orbit		
19,999	−0.000001621	0.000092	1.000001621
20,000	−0.000001624	−0.000065	1.000001624
20,001	−0.000001580	−0.000222	1.000001580

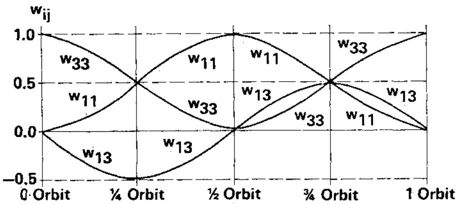

Fig. 2 Behavior of w_{11}, w_{13} and w_{33}.

The corresponding differences, as determined from Table 1, are given by

$$|\Delta w_{11}| = 9(10)^{-7} \quad (62)$$

$$|\Delta w_{33}| = 2(10)^{-6} \quad (63)$$

Thus, it may be observed that the product-variable result yields approximately one additional decimal of accuracy.

Conclusions

It is now appropriate to note some features of the product-variable system that deserve further investigation. Referring to Eq. (17), it is possible to define six redundant measures of accuracy by the quantities:

$$\Delta_{ij} = w_{ij}^2 - w_{ii}w_{jj} \quad (64)$$

where (i,j) is given by (1, 2), (1, 3), (1,4), (2, 3), (2, 4), and (3, 4). For long-term space missions, where an arithmetic failure in the computer is not "an impossible event," tolerances may be assigned to the Δ_{ij} and hence they may potentially provide an error detection and/or correction capability. More generally, one might anticipate that the Δ_{ij} quantities should play a very analogous role to orthogonality in the direction cosine reference system. In this latter system, Bar-Itzhack and Fegley[3] have obtained some marked improvements in accuracy through application of orthogonality conditions.

References

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